Regions of Behavior

The Jacobian of the predator-prey system is

There are four main regions of behavior for the system. First, note that the [0,0] fixed point is always a saddle point (and thus unstable): the Jacobian of the system near it is

which has eigenvalues 1 and -1, regardless of μ or σ. This makes the fixed point a saddle point.

The stability and behavior of the other two points changes as a function of μ and σ. First, a Hopf Bifurcation occurs when the [1, 1-μ] changes from an unstable focus to a stable focus. This occurs when the trace of the Jacobian near that fixed point equals zero:

Therefore, a Hopf bifurcation occurs when , or, in other words, when . For values of μ below this value, every fixed point is unstable and the system exhibits a limit cycle (the biomass of predators and prey oscillates).

In addition, a transcritical bifurcation occurs when the [1, 1-μ] and the [1/μ, 0] fixed points exchange stabilities. The two fixed points coincide when μ = 1, which suggests a transcritical bifurcation. Indeed, at this point, the Jacobian is

One of the eigenvalues of this matrix is 0, so we see that a transcritical bifurcation does, in fact, occur at this point. The [1, 1-μ] fixed point exchanges stability with the [1/μ, 0] fixed point.

For values between the two bifurcations, the stable node is [1, 1-μ]. The Jacobian of the system near this point is

The determinant of this matrix is

When the determinant is positive, the system will exhibit stable node behavior. When it is negative, the system will exhibit stable focus behavior. The determinant switches sign when

Solving for μ,

For values of μ larger than this value, the fixed point will be a stable node. For values below this value, the fixed point will be a stable focus.

In summary, then, for values of μ less than (the Hopf Bifurcation), the system is entirely unstable and exhibits a limit cycle. For values of μ greater than 1, the [1/μ, 0] fixed point is a stable node and the other two fixed points are unstable. For values of Μ between the the Hopf Bifurcation and 1, the [1, 1-μ] fixed point is stable. It is a stable focus for , and a stable node for . These behaviors are reflected in the parameter space plot in our GUI.